NRL REPORT 3686

ELLIPTICALLY POLARIZED WAVES AND ANTENNAS

M. L. Kales and J. I. Bohnert

June 22, 1950

Approved by:

Dr. L. C. Van Atta, Head, Antenna Research Branch
Dr. J. M. Miller, Superintendent, Radio Division I

NAVAL RESEARCH LABORATORY
CAPTAIN F. R. FURTH, USN, DIRECTOR
WASHINGTON, D.C.

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **22 JUN 1950**	2. REPORT TYPE **N/A**	3. DATES COVERED
4. TITLE AND SUBTITLE **Elliptically Polarized Waves and Antennas**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Naval Research Laboratory, Washington, D.C. 20375-5335**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release, distribution unlimited.**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**31**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

DISTRIBUTION

ONR
Attn: Mr. H. Harrison 1

CNO 1

BuAer
Attn: Mr. E. P. McDowell 1

BuOrd
Attn: Mr. A. D. Bartelt 1

BuShips
Attn: Mr. Alfred W. Andrews 1

CO & Dir., USNEL
Attn: Dr. T. J. Keary 1
Attn: Dr. F. R. Abbott 1

CO & Dir., USNUSL
Attn: Mr. C. M. Dunn 1

CDR, USNPG 1

CDR, NADC, Aeronautical Electronic & Electrical Lab.
Attn: Dr. H. Krutter 1
Attn: Mr. E. Steele 1
Attn: Mr. E. R. Schlieben 1

CO, NATS, Patuxent River
Attn: Mr. L. S. Marquardt 1

OCSigO
Attn: Mr. J. J. Kelleher 1
Attn: Mr. A. R. Beach 1

Dir., ESL
Attn: Mr. O. C. Woodyard 1
Attn: Mr. Leonard Moore 1

Ch. of Staff, USAF
Attn: Mr. Joseph Weichbrod, Room 4D-112 1

Wright-Patterson AFB	
Attn: BAU-CADO	1
Attn: CADO-E1	2
Attn: Mr. T. J. Gibbons (MCREEO)	1
CO, Air Force Cambridge Research Labs.,	
Attn: Dr. R. C. Spencer	1
Attn: Mr. R. M. Barrett	1
Div. of Public Documents, G.P.O.	
Attn: Library	1
Office of Tech. Services, Dept. of Commerce	2
Dir., NBS	
Central Radio Propagation Laboratory, Microwave Standards Section	
Attn: Dr. Harold Lyons	1
Attn: Dr. Harold Goldberg	1
Attn: Dr. A. G. McNish	1
RDB	
Attn: Mr. M. Barry Carlton, Committee on Electronics	1
Attn: Information Requirements Branch	2
Attn: Navy Secretary	1
Naval Res. Sec., Science Div., Library of Congress	
Attn: Mr. J. H. Heald	2

CONTENTS

ABSTRACT

Various properties of elliptically polarized waves and antennas are investigated by utilizing a complex vector method. Well-known relations derived for linear polarization appear as special cases of more general relations for elliptical polarization. A geometrical representation of complex vectors provides a corresponding geometrical interpretation of many of the results obtained. Elliptically polarized fields are resolved into "orthogonal" elliptically polarized components, of which linear and circular components are special cases.

Formulas are obtained relating to the polarization ellipse, the polarization pattern, ellipticity, phase, and sense of polarization. A unit complex polarization vector and complex transmission and reception vectors are introduced to represent respectively the state of polarization of a given field, the field transmitted, and the voltage received. With the aid of these vectors, the transmitting and receiving properties of antennas are obtained.

The notions of radiation pattern, gain, beamwidth, and phase are examined. Measuring techniques involving linearly polarized and circularly polarized field components are discussed and compared for accuracy. A power transfer equation between two elliptically polarized antennas is obtained in terms of the gains and polarization characteristics of the two antennas, and is applied to special cases.

PROBLEM STATUS

This report concludes the work on this phase of the problem. Work continues on the basic problem.

AUTHORIZATION

NRL Problem No. R09-31R
NR 509-310

ELLIPTICALLY POLARIZED WAVES AND ANTENNAS

INTRODUCTION

Recently there has been considerable interest in elliptically polarized waves and antennas. Various aspects of the subject have been treated independently by members of the Antenna Research Branch and by others.[1] The principal purpose of this report is to make available a comprehensive and unified treatment of the subject. In addition, many new results not previously published are presented.

In an effort to avoid ambiguities, complex vectors and scalars are represented by script symbols, and real vectors and scalars by print symbols. Departures from this notation occur only when there is a conflict with generally accepted notation.

The report is divided into two parts, the first devoted to theory and the second to measurements.

A. THEORY

General Principles--Complex Vector Representation of Elliptically Polarized Fields

A uniform elliptically polarized plane wave may be defined as a wave of the form

$$\vec{\mathcal{E}}(t) = \vec{\mathcal{E}} e^{j(\omega t - kz)} ,$$

where $\vec{\mathcal{E}}$ is a complex vector of the form

$$\vec{\mathcal{E}} = \vec{E}_r + j\vec{E}_i , \tag{1}$$

and $\vec{E}_r$ and $\vec{E}_i$ are real vectors perpendicular to the z-axis and independent of position and time. Thus a uniform elliptically polarized wave is obtained by the superposition of two uniform linearly polarized plane waves, which are in phase quadrature and which are travelling in the same direction. There is clearly no loss of generality in assuming that the direction of propagation is in the direction of the z-axis.

---

[1] A joint paper by the authors and others to appear in the Proceedings of the I.R.E. will provide the reader with additional information and with other points of view.

It can readily be shown that the wave $\vec{\mathcal{E}}(t)$ can be expressed in the form

$$\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}' e^{j(\omega t - kz + \delta)} \quad , \tag{2}$$

where $\vec{\mathcal{E}}' = \vec{E}_r' + j\vec{E}_i'$ and $\vec{E}_r' \cdot \vec{E}_i' = 0$. The angle δ and the vectors $\vec{E}_r'$ and $\vec{E}_i'$ are determined by the equations

$$\tan 2\delta = 2\vec{E}_r \cdot \vec{E}_i \Big/ (\vec{E}_r^2 - \vec{E}_i^2) \quad ,$$

$$\vec{E}_r' = \vec{E}_r \cos\delta + \vec{E}_i \sin\delta \quad ,$$

$$\vec{E}_i' = -\vec{E}_r \sin\delta + \vec{E}_i \cos\delta \quad .$$

Thus an elliptically polarized wave can also be regarded as the superposition of two linearly polarized waves which are in both time and space quadrature.

To see why such a wave is described as elliptically polarized, consider the locus of the termini of the instantaneous $\vec{E}$-vectors drawn through a given point in space, where $\vec{E} = \mathrm{Re}\,\vec{\mathcal{E}}(t)$. Since $\vec{E}_r'$ and $\vec{E}_i'$ are orthogonal, we can choose orthogonal x- and y-axes through the given point and in the directions of $\vec{E}_r'$ and $\vec{E}_i'$ respectively. Then we may write $\vec{E}_r' = E_x \vec{i}_x$ and $\vec{E}_i' = E_y \vec{i}_y$, where $\vec{i}_x$ and $\vec{i}_y$ are unit vectors in the directions of the x- and y-axes. Equation (2) may be expressed in the form

$$\vec{\mathcal{E}}(t) = (E_x \vec{i}_x + j\, E_y \vec{i}_y) e^{j(\omega t + \delta)} \quad ; \tag{3}$$

then

$$\vec{E} = E_x \cos(\omega t + \delta)\vec{i}_x - E_y \sin(\omega t + \delta)\vec{i}_y \quad .$$

The coordinates of the terminus of $\vec{E}$ are given by $x = E_x \cos(\omega t + \delta)$ and $y = -E_y \sin(\omega t + \delta)$, so that

$$x^2/E_x^2 + y^2/E_y^2 = 1 \quad .$$

Thus the locus is an ellipse with axes in the directions of $\vec{E}_r'$ and $\vec{E}_i'$, and with lengths of semi-axes of $|\vec{E}_r'|$ and $|\vec{E}_i'|$. This ellipse is called the polarization ellipse. The ratio of the major to the minor axis is called the ellipticity of polarization.

Since in considering the locus of the terminus of the $\vec{E}$-vector at a given point only the variation with time was involved, it follows that if at any given point in space an arbitrary vector function of the time is given by

$$\vec{\mathcal{U}}(t) = \vec{\mathcal{U}} e^{j\omega t} = (\vec{U}_r + j\vec{U}_i)e^{j\omega t} \quad ; \tag{4}$$

then the locus of the terminus of the vector which is the $\mathrm{Re}\,\vec{\mathcal{U}}(t) = \vec{U}(t)$ is an ellipse lying in the plane of $\vec{U}_r$ and $\vec{U}_i$. In passing, it might be pointed out that any vector of the form

$$\vec{U} = U_x \cos(\delta_x + \omega t)\vec{i}_x + U_y \cos(\delta_y + \omega t)\vec{i}_y + U_z \cos(\delta_z + \omega t)\vec{i}_z$$

describes an ellipse, since such a vector may be expressed in the form of Equation (4) as

$$\vec{U} = \mathrm{Re}\Big[\Big\{U_x \cos\delta_x \vec{i}_x + U_y \cos\delta_y \vec{i}_y + U_z \cos\delta_z \vec{i}_z\Big\} + j\Big\{U_x \sin\delta_x \vec{i}_x + U_y \sin\delta_y \vec{i}_y + U_z \sin\delta_z \vec{i}_z\Big\}\Big]e^{j\omega t} \quad .$$

a positive normal is assigned to the plane of the polarization ellipse of $\vec{U}(t)$, the ...ization is defined as right-handed or left-handed according as the vector is rotating ...wise or counterclockwise when viewed by an observer looking in the direction of the ...al. In the case of an elliptically polarized plane wave, the positive normal is usual... ...ken in the direction of propagation.

If an arbitrary pair of orthogonal unit vectors $\vec{i}_x$ and $\vec{i}_y$ are selected in the plane of ..., then $\vec{\mathcal{U}}(t)$ may be expressed in the form

$$\vec{\mathcal{U}}(t) = (\mathcal{U}_x \vec{i}_x + \mathcal{U}_y \vec{i}_y)e^{j\omega t}, \tag{5}$$

...e $\mathcal{U}_x$ and $\mathcal{U}_y$ are complex. Since an elliptically polarized wave is often expressed in ...above form, we shall derive a criterion for determining the sense of rotation of $\vec{U}(t)$. ...$\mathcal{U}_x = |\mathcal{U}_x| e^{j\delta_x}$ and $\mathcal{U}_y = |\mathcal{U}_y| e^{j\delta_y}$; then

$$\vec{U}(t) = |\mathcal{U}_x| \cos(\omega t + \delta_x)\vec{i}_x + |\mathcal{U}_y| \cos(\omega t + \delta_y)\vec{i}_y .$$

...denote the angle that $\vec{U}(t)$ makes with $\vec{i}_x$ and let the positive direction of θ be from ...$\vec{i}_y$. Then

$$\tan\theta = \left|\frac{\mathcal{U}_y}{\mathcal{U}_x}\right| \frac{\cos(\omega t + \delta_y)}{\cos(\omega t + \delta_x)}$$

$$= \left|\frac{\mathcal{U}_y}{\mathcal{U}_x}\right| \left\{\cos(\delta_y - \delta_x) + \sin(\delta_x - \delta_y) \tan(\omega t + \delta_x)\right\},$$

and...

$$\sec^2\theta \frac{d\theta}{dt} = \omega \left|\frac{\mathcal{U}_y}{\mathcal{U}_x}\right| \sin(\delta_x - \delta_y) \sec^2(\omega t + \delta_x).$$

Thus it is seen that $\frac{d\theta}{dt}$ is positive or negative according as $\sin(\delta_x - \delta_y)$ is positive or negative. If the unit normal $\vec{n}$ to the plane of $\vec{U}(t)$ is chosen in such a way that the vectors $\vec{i}_x$, $\vec{i}_y$, $\vec{n}$ form a right-handed system when taken in the indicated order, then the polarization is right-handed or left-handed according as $\sin(\delta_x - \delta_y)$ is positive or negative, or in other words, according as the x-component of $\vec{U}(t)$ leads or lags the y-component.

Algebra of Complex Vectors

The use of complex vectors in representing elliptically polarized waves suggests the possibility of using a complex vector algebra in studying such waves. An appropriate algebra, which will be used in the subsequent discussion, will now be outlined.

A complex vector is defined as a vector of the form

$$\vec{\mathcal{U}} = \vec{U}_r + j\vec{U}_i ,$$

$\vec{U}_r$ and $\vec{U}_i$ being real vectors which in general are not in the same direction. As has been seen, at a given point in space, $\vec{U}(t) = \mathrm{Re}(\vec{U}_r + j\vec{U}_i)e^{j\omega t}$ describes an ellipse. The direction of rotation of $\vec{U}(t)$ may be indicated by means of an arrow placed on the ellipse. Such a directed ellipse may be regarded as a geometrical representation of a complex vector. It should be noted that a complex vector is not simply the product of a real vector and a complex scalar; this will be true only if $\vec{U}_r$ and $\vec{U}_i$ are in the same direction, in which case $\vec{\mathcal{U}}$ is linearly polarized.

In general, a complex vector determines a plane, namely the plane of $\vec{U}_r$ and $\vec{U}_i$. In the discussion which follows it is assumed that the complex vectors under consideration lie in parallel planes unless otherwise indicated.

The scalar product of two complex vectors, $\vec{\mathcal{U}} = \vec{U}_r + j\vec{U}_i$ and $\vec{\mathcal{V}} = \vec{V}_r + j\vec{V}_i$ is defined as

$$\vec{\mathcal{U}} \cdot \vec{\mathcal{V}} = (\vec{U}_r \cdot \vec{V}_r - \vec{U}_i \cdot \vec{V}_i) + j(\vec{U}_i \cdot \vec{V}_r + \vec{U}_r \cdot \vec{V}_i).$$

Their vector product is defined as

$$\vec{\mathcal{U}} \times \vec{\mathcal{V}} = (\vec{U}_r \times \vec{V}_r - \vec{U}_i \times \vec{V}_i) + j(\vec{U}_r \times \vec{V}_i + \vec{U}_i \times \vec{V}_r).$$

The magnitude of $\vec{\mathcal{U}}$ is $|\vec{\mathcal{U}}| = \sqrt{\vec{\mathcal{U}} \cdot \vec{\mathcal{U}}^*}$. Two vectors, $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$, are said to be orthogonal if $\vec{\mathcal{U}} \cdot \vec{\mathcal{V}}^* = 0$, and parallel if $\vec{\mathcal{U}} \times \vec{\mathcal{V}} = 0$.

It is not difficult to show that the conditions of orthogonality and parallelism are equivalent respectively to the two relations $\vec{\mathcal{V}} = \kappa_1 \vec{\mathcal{U}}^* \times \vec{n}$ and $\vec{\mathcal{V}} = \kappa_2 \vec{\mathcal{U}}$, where the κ's are constants. These relations have an interesting geometrical interpretation. As in Equation (2), $\vec{\mathcal{U}} e^{j\omega t}$ may be expressed in the form $\vec{\mathcal{U}} e^{j\omega t} = (\vec{U}_r' + j\vec{U}_i')\, e^{j(\omega t + \delta)}$, where $\vec{U}_r'$ and $\vec{U}_i'$ are a pair of orthogonal real vectors in the plane of $\vec{U}$. From this form, and the criterion for sense of rotation, it is at once evident that the ellipse determined by $\vec{\mathcal{U}}^* e^{j\omega t} = (\vec{U}_r' - j\vec{U}_i')\, e^{j(\omega t - \delta)}$ has the same orientation and ellipticity as that determined by $\vec{\mathcal{U}} e^{j\omega t}$, but has the opposite sense. It is also clear that $\vec{\mathcal{U}}^* \times \vec{n}$ is obtained by rotating $\vec{\mathcal{U}}$ through 90° in the plane of $\vec{\mathcal{U}}$. Thus two complex vectors $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ are orthogonal if and only if the ellipses which represent them have the same ellipticity, opposite sense of rotation, and major axes perpendicular. From the relation which expresses the condition of parallelism it is seen that $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ are parallel if and only if the ellipses representing them have the same orientation, ellipticity, and sense of rotation.

Resolution of Elliptically Polarized Fields into Orthogonal Elliptically Polarized Components

If $\vec{\mathcal{E}}$ is an arbitrary complex vector, and if $\vec{\mu}$ and $\vec{\nu}$ denote an arbitrary pair of complex orthogonal unit vectors in the plane of $\vec{\mathcal{E}}$, then $\vec{\mathcal{E}}$ may be represented in the form

$$\left.\begin{aligned} \vec{\mathcal{E}} &= (\vec{\mathcal{E}} \cdot \vec{\mu}^*)\vec{\mu} + (\vec{\mathcal{E}} \cdot \vec{\nu}^*)\vec{\nu} \\ &= \mathcal{E}_\mu \vec{\mu} + \mathcal{E}_\nu \vec{\nu} \end{aligned}\right\} . \tag{6}$$

From the orthogonality of $\vec{\mu}$ and $\vec{\nu}$, it follows that

$$|\vec{\mathcal{E}}|^2 = |\mathcal{E}_\mu|^2 + |\mathcal{E}_\nu|^2 . \tag{7}$$

From Equation (6) we conclude that a given elliptically polarized field can be resolved into two orthogonal elliptically polarized fields in an infinite number of ways. The ellipticity, orientation, and sense of the ellipse representing one of the component fields can be arbitrarily specified, in which case the other component field will be represented by an ellipse having the same ellipticity, opposite sense, and having its major axis perpendicular to the major axis of the first. Equation (7) states that the power density in an elliptically polarized wave is the sum of the power densities of any two orthogonal, elliptically polarized components.

Consider several special cases of Equation (6). If $\vec{\mathfrak{u}}$ and $\vec{\mathfrak{v}}$ are real, then the component fields represented by $\mathcal{E}_u \vec{\mathfrak{u}}$ and $\mathcal{E}_v \vec{\mathfrak{v}}$ are linearly polarized. It follows from Equation (3) that $\vec{\mathcal{E}}$ may be expressed in the form $\vec{\mathcal{E}} = (E_x \vec{i}_x + j E_y \vec{i}_y)e^{j\delta}$. Now let $\vec{u}$ and $\vec{v}$ be a pair of real unit orthogonal vectors making an angle ψ with $\vec{i}_x$ and $\vec{i}_y$ respectively. Then

$$\mathcal{E}_u = (E_x \cos\psi + j E_y \sin\psi)e^{j\delta} \quad ,$$

$$\mathcal{E}_v = (-E_x \sin\psi + j E_y \cos\psi)e^{j\delta} \quad .$$

If ψ is chosen so that $|\mathcal{E}_u|^2 = |\mathcal{E}_v|^2$, then $(E_x^2 - E_y^2)\cos 2\psi = 0$. Thus $|\mathcal{E}_u| = |\mathcal{E}_v|$ if $\psi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ as is obvious from the geometry of the polarization ellipse.

From the above results we conclude that an elliptically polarized field of arbitrary ellipticity and sense of rotation can be produced by superimposing two orthogonal linearly polarized fields for which either (a) the two components are in phase quadrature and the amplitudes are properly chosen, or (b) the two components are equal in amplitude and the phase is properly chosen. In the first case, the axes of the ellipse will be in the directions of the two components which are in phase quadrature, and in the second case, the axes of the ellipse will bisect the angles between the two components of equal amplitude.

The resolution of an elliptically polarized field into right- and left-handed circularly polarized components is of considerable importance. Let $\vec{\mathcal{E}}$ be represented in the form of Equation (5),

$$\vec{\mathcal{E}} = \mathcal{E}_x \vec{i}_x + \mathcal{E}_y \vec{i}_y \quad . \tag{8}$$

A pair of right- and left-handed circularly polarized unit vectors may be selected as

$$\vec{\mathfrak{u}}_R = (\vec{i}_x - j\,\vec{i}_y)/\sqrt{2}$$

and

$$\vec{\mathfrak{u}}_L = (\vec{i}_x + j\,\vec{i}_y)/\sqrt{2} \quad .$$

Applying Equation (6), Equation (8) may be written

$$\vec{\mathcal{E}} = \frac{1}{2}(\mathcal{E}_x + j\mathcal{E}_y)(\vec{i}_x - j\,\vec{i}_y) + \frac{1}{2}(\mathcal{E}_x - j\mathcal{E}_y)(\vec{i}_x + j\,\vec{i}_y)$$

$$= \mathcal{E}_R \vec{\mathfrak{u}}_R + \mathcal{E}_L \vec{\mathfrak{u}}_L \quad . \tag{9}$$

From the discussion of the conditions for orthogonality and parallelism it is evident that a pair of right- and left-handed circularly polarized vectors are always orthogonal and two circularly polarized vectors of the same sense are parallel, provided the same unit normal $\vec{n}$ is used in determining the sense of polarization. Hence if $\vec{\mathfrak{u}}'_R$ and $\vec{\mathfrak{u}}'_L$ denote any other pair of right- and left-handed circularly polarized unit vectors, then $\vec{\mathfrak{u}}'_R = \vec{\mathfrak{u}}_R e^{j\delta}$ and $\vec{\mathfrak{u}}'_L = \vec{\mathfrak{u}}_L e^{j\delta'}$. From Equation (6) we see that $\vec{\mathcal{E}} = (\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}'^*_R)\vec{\mathfrak{u}}'_R + (\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}'^*_L)\vec{\mathfrak{u}}'_L$. But $(\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}'^*_R)\vec{\mathfrak{u}}'_R = (\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}^*_R e^{-j\delta})e^{j\delta}\vec{\mathfrak{u}}_R = (\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}^*_R)\vec{\mathfrak{u}}_R$, and similarly $(\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}'^*_L)\vec{\mathfrak{u}}'_L = (\vec{\mathcal{E}} \cdot \vec{\mathfrak{u}}^*_L)\vec{\mathfrak{u}}_L$. Thus the component vectors on the right of Equation (9) are independent of the particular choice of right- and left-handed circularly polarized unit vectors, or in other words, the resolution into right- and left-handed circularly polarized fields is unique. It is clear also that $|\mathcal{E}'_R| = |\mathcal{E}_R|$ and $|\mathcal{E}'_L| = |\mathcal{E}_L|$.

Since the vectors which are the real parts of $\mathcal{E}_R \vec{u}_R e^{j\omega t}$ and $\mathcal{E}_L \vec{u}_L e^{j\omega t}$ rotate in opposite directions with constant magnitudes $|\mathcal{E}_R|$ and $|\mathcal{E}_L|$, it is seen at once that $|\vec{E}|_{max} = |\mathcal{E}_R| + |\mathcal{E}_L|$ and $|\vec{E}|_{min} = \| \mathcal{E}_R| - |\mathcal{E}_L\|$. Hence the ellipticity of polarization, e, is given by

$$e = \| \mathcal{E}_R| + |\mathcal{E}_L| / |\mathcal{E}_R| - |\mathcal{E}_L \| . \qquad (10)$$

It is easy to show that $\vec{\mathcal{E}}$ is right- or left-handed according as $|\mathcal{E}_R|$ is greater or less than $|\mathcal{E}_L|$. It is only necessary to observe that unit vectors $\vec{i}_x$ and $\vec{i}_y$ can be determined so that $\vec{\mathcal{E}}$ takes the form of Equation (3), $\vec{\mathcal{E}} = (E_x \vec{i}_x - j E_y \vec{i}_y) e^{j\delta}$, where $E_x \geq |E_y|$. Then from the criterion for sense of rotation, $\vec{\mathcal{E}}$ will be right- or left-handed as E_y is positive or negative. Since $|\mathcal{E}_R| = \frac{1}{\sqrt{2}} |E_x + E_y|$ and $|\mathcal{E}_L| = \frac{1}{\sqrt{2}} |E_x - E_y|$, E_y is positive or negative according as $|\mathcal{E}_R|$ is greater or less than $|\mathcal{E}_L|$. Thus both the ellipticity and sense of rotation of an elliptically polarized field are determined when the magnitude of the right- and left-handed circularly polarized components are known.

It is interesting to note that whenever a complex unit vector $\vec{u}$ determines a circularly polarized vector, then the vector determined by $\vec{u} e^{j\delta}$ is obtained by rotating the former through an angle δ in the direction in which it is rotating. This is apparent when one considers that a circularly polarized vector rotates with constant angular velocity ω.

Significance of the Term Phase Applied to Elliptically Polarized Fields

Although the primary concern in this paper is with elliptically polarized fields and antennas, it seems appropriate at this point to consider the question of what is meant by phase. Some confusion exists which seems to stem from the fact that the term phase was defined originally in connection with scalar quantities, but has been used without adequate definition in dealing with vector quantities. One method that has been suggested for treating this subject is to consider the phase of the components of a vector field, since the components are scalar quantities. While such an approach would be perfectly valid, it would make the phase dependent on the coordinate system, and in general such a definition would have doubtful physical significance. Another approach, which the authors favor, is to refer the field at each point in space to a unit vector in the direction of the

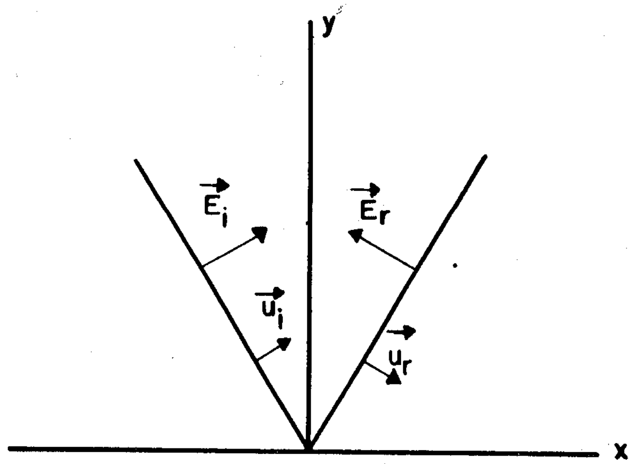

Figure 1 - Phase change upon reflection

field. Thus if $\vec{u}$ is a unit vector in the direction of $\vec{\mathcal{E}}$, and we write $\vec{\mathcal{E}} = \mathcal{E}_u \vec{u}$, then the phase of $\vec{\mathcal{E}}$ may be defined as the argument of the complex amplitude $\mathcal{E}_u$.

A simple illustration may help to make these points clear. For example, one often hears the statement made that if a uniform plane wave is incident on a perfectly conducting infinite plane at an oblique angle, then at every point of the plane the incident and reflected $\vec{E}$-vectors are 180° out of phase. This statement is true only if properly interpreted. For the sake of simplicity let us assume the $\vec{E}$-vector is in the plane of incidence. If the incident and reflected waves are referred to unit vectors $\vec{u}_i$ and $\vec{u}_r$ as shown in Figure 1, then it is true that the incident and reflected $\vec{E}$-vectors differ in phase by 180° at the points of reflection. On the other hand, if one considers the phases of the x- and y-components, then the x-components are 180° out of phase, but the y-components are in phase upon reflection.

In the case of linearly polarized fields, if the suggested definition of phase is employed, there will be an ambiguity of 180° in the determination of the phase, which must be resolved by some other means. For if $\vec{u}$ is a real unit vector in the direction of $\vec{E}$, then so is $-\vec{u}$. In the case of elliptically polarized fields the ambiguity is even greater. For if $\vec{u}$ is any complex unit vector, then $\vec{u}' = \vec{u} e^{j\delta}$ is parallel to $\vec{u}$ for an arbitrary δ. This means that the phase of $\vec{\mathcal{E}}$ can have an infinite number of distinct values, depending on the choice of $\vec{u}$. By a suitable choice of $\vec{u}$ it is possible, however, to reduce the ambiguity to one of 180°. If we let $\vec{i}_x$ and $\vec{i}_y$ be in the directions of the major and minor axes respectively of the ellipse which represents $\vec{\mathcal{E}}$, then a suitable vector $\vec{u}$ may be defined with the aid of a parameter, α, by the equation

$$\vec{u} = \sin\alpha\ \vec{i}_x - j\cos\alpha\ \vec{i}_y \quad . \tag{11}$$

We can then write $\vec{\mathcal{E}} = |\vec{\mathcal{E}}|\,\vec{u}\, e^{j\delta}$, and define δ to be the phase of $\vec{\mathcal{E}}$. Physically, this definition of phase leads to the result that if two elliptically polarized fields represented by $\vec{\mathcal{E}}_1$ and $\vec{\mathcal{E}}_2$ are in phase, the real vectors $\vec{E}_1$ and $\vec{E}_2$ assume their maximum and minimum lengths at the same instants of time. On the other hand if $\vec{\mathcal{E}}_1$ and $\vec{\mathcal{E}}_2$ differ in phase by 90°, the length of one is a maximum whenever the length of the other is a minimum. For phase differences other than a multiple of 90° the situation is more complicated. It should be noted that the definition of phase that has been suggested applies equally well to linearly and elliptically polarized fields. For fields that are circularly polarized this definition of phase still leads to ambiguous results since in this case the vectors $\vec{i}_x$ and $\vec{i}_y$ can be chosen in an infinite number of ways.

For future reference it may be noted that when $\vec{u}$ is expressed in the form of Equation (11), α can always be chosen in the interval $\frac{\pi}{4} \leq \alpha \leq \frac{3\pi}{4}$, and $e = |\tan\alpha|$. If $\vec{i}_x$, $\vec{i}_y$, $\vec{i}$ form a right-handed system, then $\vec{u}$ is right- or left-hand polarized according as α is in the first or second quadrant.

Miscellaneous Relations

Before turning to the subject of antennas, a few relations of interest in connection with measurements will be derived. Assume that a dipole is rotated in the plane of polarization of an elliptically polarized field. Let a pair of orthogonal real unit vectors $\vec{i}_x$ and $\vec{i}_y$ be chosen arbitrarily in this plane. Then by a proper choice of the origin of time $\vec{\mathcal{E}}$ may be expressed in the form $\vec{\mathcal{E}} = E_x \vec{i}_x e^{j\delta} + E_y \vec{i}_y$, where E_x and E_y are positive. Let the dipole make an angle ψ with $\vec{i}_x$; then the component of $\vec{\mathcal{E}}$ in the direction of the dipole is $\mathcal{E}_\psi = E_x \cos\psi\, e^{j\delta} + E_y \sin\psi$. The power received by the dipole is proportional to

$$\left.\begin{aligned} |\mathcal{E}_\psi|^2 &= E_x^2 \cos^2\psi + E_y^2 \sin^2\psi + 2\, E_x E_y \sin\psi \cos\psi \cos\delta \\ &= \frac{1}{2}\left\{E_x^2 + E_y^2 + (E_x^2 - E_y^2)\cos 2\psi + 2\, E_x E_y \cos\delta \sin 2\psi\right\} \\ &= \frac{1}{2}\left[E_x^2 + E_y^2 + \left\{(E_x^2 + E_y^2)^2 - 4\, E_x^2 E_y^2 \sin^2\delta\right\}^{\frac{1}{2}} \cos(2\psi - \beta)\right] \end{aligned}\right\}, \qquad (12)$$

where

$$\tan\beta = 2\, E_x E_y \cos\delta / (E_x^2 - E_y^2).$$

Thus

$$e = |\mathcal{E}_\psi|_{max} \Big/ |\mathcal{E}_\psi|_{min}$$

$$e = \left[(1 + \cos\eta)/(1 - \cos\eta)\right]^{\frac{1}{2}} = \cot\frac{\eta}{2} \qquad , \qquad (13)$$

where

$$\sin\eta = \left|\, 2\, E_x E_y \sin\delta \,/\, (E_x^2 + E_y^2)\,\right| \quad .$$

In the special case where $E_x = E_y$, Equation (13) becomes $e = \max\left\{\left|\tan\frac{\delta}{2}\right|, \left|\cot\frac{\delta}{2}\right|\right\}$.

The use of Equation (13) for determining the ellipticity involves the phase difference δ between the two components. A direct measurement of δ may be avoided, however, either by measuring the maximum and minimum components or by measuring any three component amplitudes. For example, if one substitutes $\psi = \pi/4$ in the second of Equations (12), the value of $2\, E_x E_y \cos\delta$ and hence the value of $(2\, E_x E_y \sin\delta)$ may be obtained.

Equation (13) determines the ellipticity when the two given components of the field are in space quadrature but differ in phase by an angle δ. It may happen that the field is expressed in the form of Equation (1), $\vec{\mathcal{E}} = \vec{E}_r + j\,\vec{E}_i$. In this case we have two components which are in time quadrature. If we denote by δ^1 the angle between the vectors $\vec{E}_r$ and $\vec{E}_i$, then it can be shown that Equation (13) can be used to determine the ellipticity simply by replacing E_x and E_y by $|\vec{E}_r|$ and $|\vec{E}_i|$ respectively.

From Equations (12) it is evident that the angle ψ for which $|\mathcal{E}_\psi|$ is a maximum, satisfies the equation $\tan 2\psi = \tan\beta = 2\, E_x E_y \cos\delta/(E_x^2 - E_y^2)$. In order for $|\mathcal{E}_\psi|$ to be a maximum, ψ must be chosen so that one of the two inequalities $(E_x^2 - E_y^2)\cos 2\psi > 0$, or $E_x E_y \cos\delta \sin 2\psi > 0$ is satisfied. These relations determine the angle ψ which the major axis of the ellipse makes with $\vec{i}_x$. In the event that $E_x^2 = E_y^2$ and $\cos\delta = 0$, then the field is circularly polarized and $|\mathcal{E}_\psi|$ is constant. If the relation between $|\mathcal{E}_\psi|^2$ and ψ given in Equations (12) is plotted in polar coordinates, a dumbbell shaped curve is obtained which represents the power received by the dipole as a function of the angle of orientation, ψ. This curve is called a polarization pattern.

The polarization pattern may be obtained from the polarization ellipse by means of the geometrical construction shown in Figure 2. The instantaneous magnitude of the voltage received by the dipole is proportional to the projection of the rotating $\vec{E}$-vector upon a line parallel to the direction ψ of the dipole. Referring to Figure 2, let Q be the point on line PQ such that the normal to PQ at Q is tangent to the ellipse. It is evident that the maximum value of E_ψ, regarded as a function of time, is represented by PQ. Since the rms value of the voltage is proportional to the maximum value of E_ψ, the length PQ can equally well be used to represent the rms value of the voltage. The locus of the point Q is a dumbbell shaped curve which coincides with the polarization ellipse at the ends of its diameters. The polarization pattern is obtained by plotting the square of the radius PQ as a function of ψ. It is also dumbbell shaped for all values of ellipticity greater than unity. It should be noted that the polarization pattern is not obtained by squaring the corresponding radii of the polarization ellipse. Notice, however, that the ratio of the maximum value to the minimum value of the polarization pattern is equal to the square of the ratio of the major axis to the minor axis of the polarization ellipse. Hence measured values of the ellipticity e are frequently taken from polarization pattern measurements. (See Figure 4.)

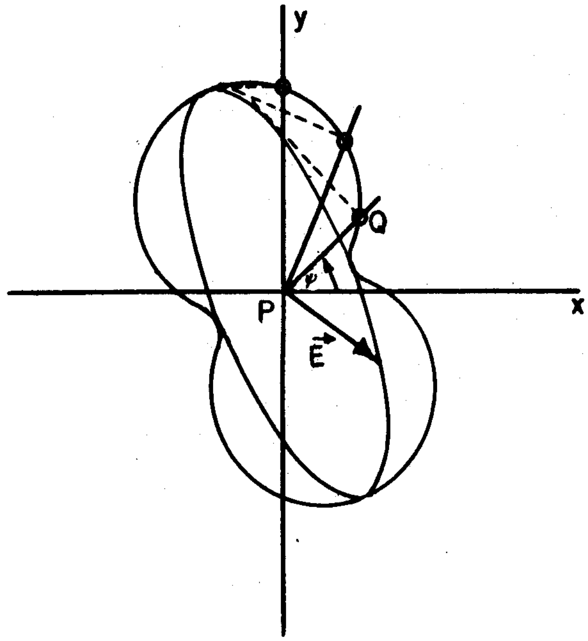

Figure 2 - Geometrical relation between the curve of received voltage and the polarization ellipse

Relations Involving Elliptically Polarized Transmitting and Receiving Antennas

Let us now consider the relation of elliptically polarized fields to antennas. The distant field $\vec{\mathcal{E}}(P, t)$ produced by an antenna can be expressed in the form

$$\vec{\mathcal{E}}(P, t) = \frac{\vec{\mathcal{E}}(\theta, \phi)}{R} e^{j(\omega t - kR)} \quad , \tag{14}$$

where $\vec{\mathcal{E}}(\theta, \phi)$ is a complex vector, and R, θ, and ϕ are the spherical coordinates of the point P. If the factor 1/R is omitted, the variation of the remaining factor along a radius is identical with that of an elliptically polarized plane wave travelling in the (θ, ϕ) direction. It is convenient to introduce a complex unit vector $\vec{p} = \vec{p}(\theta, \phi)$ defined by the relation

$$\vec{p}(\theta, \phi) = \vec{\mathcal{E}}(\theta, \phi) \Big/ \left|\vec{\mathcal{E}}(\theta, \phi)\right| \quad . \tag{15}$$

From the foregoing discussion it is clear that $\vec{p}$ completely describes the state of polarization of the field, that is, the ellipticity, sense, and orientation of the ellipse which represents the instantaneous $\vec{E}$-vector. For this reason $\vec{p}$ shall be referred to as the polarization unit vector.

Another vector which will be useful in describing the transmitted field of an antenna is

$$\vec{g}(\theta,\phi) = \sqrt{G(\theta,\phi)}\ \vec{p}(\theta,\phi) = \left(2\pi\sqrt{\epsilon/\mu}\ /P_t\right)^{\frac{1}{2}} |\mathcal{E}(\theta,\phi)|\ \vec{p}(\theta,\phi) \quad , \tag{16}$$

where $G(\theta,\phi)$ is the power gain of the antenna in the direction (θ,ϕ), and P_t is the transmitted power. This vector will be referred to as the transmission vector of the antenna. The significance of $\vec{g}$ becomes apparent when, with the aid of Equations (15) and (16), Equation (14) is rewritten in the form

$$\vec{\mathcal{E}}(P,t) = (\sqrt{\mu/\epsilon}\ /2\pi)^{\frac{1}{2}}\sqrt{P_t}\ \vec{g}\frac{e^{j(\omega t - kR)}}{R} \quad . \tag{17}$$

Since for a lossless antenna $\sqrt{P_t}$ is proportional to the voltage at the input terminals of the antenna, it is seen that the transmission vector may be used to express the radiation field of an antenna in terms of the input voltage in much the same way that the gain may be used to express the power per unit solid angle in a given direction in terms of the power delivered to the antenna. It follows immediately from the definition of $\vec{g}$, that

$$|\vec{g}(\theta,\phi)|^2 = G(\theta,\phi) \ . \tag{18}$$

By analogy with the case of linearly polarized antennas, we shall also define a vector $\vec{h} = \vec{h}(\theta,\phi)$ by the relation

$$\vec{h}(\theta,\phi) = (\lambda/\sqrt{4\pi})\ \vec{g}^*(\theta,\phi) \quad . \tag{19}$$

This vector will be referred to as the reception vector of the antenna. If we denote by A = $A(\theta,\phi)$ the receiving cross-section of the antenna, then we see from Equation (19) that

$$|\vec{h}(\theta,\phi)|^2 = (\lambda^2/4\pi)\ G(\theta,\phi) = A(\theta,\phi) \quad . \tag{20}$$

It is shown in Appendix I that if an antenna receives power from an incident elliptically polarized plane wave $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}e^{j\omega t}$, then the complex received voltage is given by $\mathcal{V} = \mathcal{K}\vec{\mathcal{E}}\cdot\vec{h}^*$. If the receiver is matched to the line which has an impedance Z_0, and if the reflection coefficient presented to the line by the antenna is Γ, then $|\mathcal{K}| = \left\{Z_0(1 - |\Gamma|^2)\sqrt{\epsilon/\mu}\right\}^{1/2}$. The power received by the antenna is given by

$$P_r = |\mathcal{V}|^2/2\ Z_0 = \frac{1}{2}(1 - |\Gamma|^2)\sqrt{\epsilon/\mu}\quad |\vec{\mathcal{E}}\cdot\vec{h}^*|^2 \quad .$$

If both the receiver and antenna are matched to the line, then

$$P_r = \frac{1}{2}\sqrt{\epsilon/\mu}\quad |\vec{\mathcal{E}}\cdot\vec{h}^*|^2 \ .$$

Let $\vec{p}$ be the polarization unit vector of the incident field, $\vec{\mathcal{E}}$, and let $\vec{g}$ be the unit vector, $\vec{h}/|\vec{h}|$. Note that $\vec{g}$ is the conjugate of the polarization unit vector of the wave which would be transmitted by the antenna. Then the expression for P_r takes the form

$$P_r = \frac{1}{2}\ \sqrt{\epsilon/\mu}\,|\vec{\mathcal{E}}|^2\ |\vec{h}|^2\ |\vec{p}\cdot\vec{g}^*|^2 \quad .$$

As is the case with real vectors, the magnitude of the scalar product of two unit complex vectors cannot exceed unity. The factor $|\vec{p}\cdot\vec{g}^*|^2$ will be called the polarization efficiency and will be denoted by f. The factor $\frac{1}{2}\sqrt{\epsilon/\mu}\ |\vec{\mathcal{E}}|^2$ represents the power per unit area in the incident field and will be denoted by S. Substituting in the formula for P_r and remembering that $|\vec{h}|^2 = A$, we obtain the formula

$$P_r = S\,A\,f \quad . \tag{21}$$

In the case where the incident and transmitted fields are both linearly polarized the factor is given by $f = \cos^2 \psi$, where ψ is the angle between the incident and transmitted vectors. When $\psi = 0$, Equation (21) reduces to the well known formula for the maximum power received by a linearly polarized antenna, $(P_r)_{max} = SA$.

The relations above make it possible to define the polarization efficiency in physical terms. The polarization efficiency is the ratio of the power received from an incident field in a given state of polarization to the power received when the polarization of the incident field is adjusted for maximum power received, the power density in the incident field being held constant.

It is interesting to consider the foregoing relations explicitly in terms of the polarization parameters which describe the incident field and the receiving antenna. It follows from Equation (20) that the vectors which represent $\mathrm{Re}\,\vec{h}e^{j\omega t}$ and $\mathrm{Re}\,\vec{g}\,e^{j\omega t}$ are rotating in opposite directions when viewed from the same point, and that the ellipses described by these vectors have the same ellipticity and orientation. However, if the positive normal to the plane of $\vec{h}$ points toward the antenna, and the positive normal to the plane of $\vec{g}$ points away from the antenna, then the sense of polarization of $\mathrm{Re}\,\vec{h}e^{j(\omega t + kR)}$ and $\mathrm{Re}\,\vec{g}\,e^{j(\omega t - kR)}$ is the same. In other words the sense of polarization of $\mathrm{Re}\,\vec{h}e^{j(\omega t + kR)}$ is the same as that of the field which would be transmitted by the receiving antenna in the given direction.

The vector $\vec{\mathcal{E}}$ lies in a plane perpendicular to the (θ, ϕ) direction. In this plane a pair of orthogonal unit vectors $\vec{i}_x$, $\vec{i}_y$ may be chosen so that $\vec{\mathcal{E}}$ takes the form of Equation (11),

$$\vec{\mathcal{E}} = |\vec{\mathcal{E}}|\,(\sin \alpha_i\ \vec{i}_x - j\,\cos \alpha_i\ \vec{i}_y)e^{j\delta_i} \quad ,$$

and another pair of orthogonal unit vectors $\vec{i}_{x'}$, $\vec{i}_{y'}$ may be chosen so that $\vec{h}$ takes the form

$$\vec{h} = |\vec{h}|\ (\sin \alpha_r\ \vec{i}_{x'} - j\,\cos \alpha_r\ \vec{i}_{y'})e^{j\delta_r} \quad .$$

The ellipticities of $\vec{\mathcal{E}}$ and $\vec{h}$ are given by $e_i = |\tan \alpha_i|$ and $e_r = |\tan \alpha_r|$ respectively, where $\frac{\pi}{4} \leqq \begin{matrix}\alpha_i\\ \alpha_r\end{matrix} \leqq \frac{3\pi}{4}$. Using the above expressions for $\vec{\mathcal{E}}$ and $\vec{h}$, we find after some manipulation that

$$|\vec{\mathcal{E}}\cdot\vec{h}^*|^2 = \frac{1}{2}|\vec{\mathcal{E}}|^2\,|\vec{h}|^2\left\{1 + \sin 2\alpha_i\ \sin 2\alpha_r + \cos 2\alpha_i\ \cos 2\alpha_r\ \cos 2\psi\right\},$$

where ψ is the angle between $\vec{i}_{x'}$ and $\vec{i}_x$. Thus the polarization efficiency, f, is given by

$$f = \frac{1}{2}\left\{1 + \sin 2\alpha_i\ \sin 2\alpha_r + \cos 2\alpha_i\ \cos 2\alpha_r\ \cos 2\psi\right\} \quad . \tag{22}$$

In terms of the ellipticities e_i and e_r of the incident and transmitted fields, f is given by

$$f = \frac{(1 + e_i^2)(1 + e_r^2) \pm 4\,e_i e_r + (1 - e_i^2)(1 - e_r^2)\cos 2\psi}{4\,(1 + e_i^2)\ (1 + e_r^2)}\ , \tag{23}$$

where the + or - sign of the term $4\,e_i e_r$ is to be used according as the two fields have the same or opposite sense of polarization.

If the receiving antenna is rotated about an axis in the (θ, ϕ) direction, then only the angle ψ will vary, and the relative maximum and minimum values of f are attained from the Equation (22) when $\psi = 0$ and $\frac{\pi}{2}$ respectively. We thus find that

$$f_{max} = \frac{1}{2}\left\{1 + \cos 2(\alpha_i - \alpha_r)\right\} = \cos^2(\alpha_i - \alpha_r) \quad ,$$

$$f_{min} = \frac{1}{2}\left\{1 - \cos 2(\alpha_i + \alpha_r)\right\} = \sin^2(\alpha_i + \alpha_r) \quad . \tag{24}$$

If these results are interpreted in terms of the states of polarization of the incident field and of the field which would be transmitted by the receiving antenna in the given direction, it is seen that only the relative orientation of the ellipses representing the two fields is varied and that f assumes its relative maximum and minimum values when the major axes of the two ellipses are parallel and perpendicular respectively.

If sense of polarization is considered in addition to variable ψ, then both f_{max} and f_{min} of Equation (24) will have two values, the greater or lesser value according as the two senses of polarization are the same or opposite respectively.

Finally, if the orientation of the ellipses, the ellipticity, and the sense of polarization are all allowed to vary, f will have an absolute maximum value of unity when the major axes are parallel, the ellipticities are equal, and the polarizations are of the same sense; and f will have a minimum value of zero when the major axes are orthogonal, the ellipticities are equal, and the polarizations are of the opposite sense. This last case may be expressed very simply in vector language: the polarization efficiency f has an absolute maximum value of unity when $\mathcal{E}$ is parallel to $\mathcal{h}$, and a minimum value of zero when $\mathcal{E}$ is orthogonal to $\mathcal{h}$. It follows that if the incident field is resolved into two complex vector components, one of which is parallel to $\mathcal{h}$ and the other orthogonal to $\mathcal{h}$, all the power received by the antenna will be extracted from the parallel component, the orthogonal component being completely rejected. In the case of linearly polarized fields, when a particular direction of polarization is of interest, a component of the field which is orthogonal to this direction is said to be a cross-polarized component. It is clear that this notion may be extended to elliptically polarized fields, two elliptically polarized components being regarded as cross-polarized if they are orthogonal. Thus, for example, if we consider a circularly polarized field of a given sense, the circularly polarized component of the opposite sense may be regarded as the cross-polarized component.

If the factor f is plotted as a function of ψ in polar coordinates, a graph is obtained showing the variation of received power with the orientation ψ of the receiving antenna. A comparison of Equation (22) with Equations (12) shows that in both cases the functional dependence on ψ is of the same form, the two differing only in the constants involved. Thus, whether a linearly or an elliptically polarized receiving antenna is used, the variation of received power with orientation angle ψ will be represented by the same type of dumbbell shaped curve. The curves for the two cases will differ in the orientation of their axes and in the ratio of the major to minor axis.

EASURING TECHNIQUES

urement of Polarization Characteristics

To obtain a complete description of the radiation characteristics of an elliptically arized antenna, it is necessary to measure not only the distribution of radiation intenty as a function of direction but also the polarization characteristics as well. Two ethods that have been used in the Antenna Research Branch for measuring polarization aracteristics are outlined in the following paragraphs: one method employs a rotating nearly polarized antenna; the other employs two circularly polarized antennas.

Rotating Linearly Polarized Antenna - An obvious way to make measurements on an lliptically polarized radiation pattern is to explore the far field with a dipole or some ther small linearly polarized antenna. That is, for each considered direction (θ, ϕ), the power received is measured as a function of the orientation of the receiving dipole as it is rotated in the plane normal to the direction of propagation. To reduce the labor involved, a special mount and associated circuitry are used. This equipment is shown in operating condition in Figure 3. The antenna under test is used for transmitting, and the

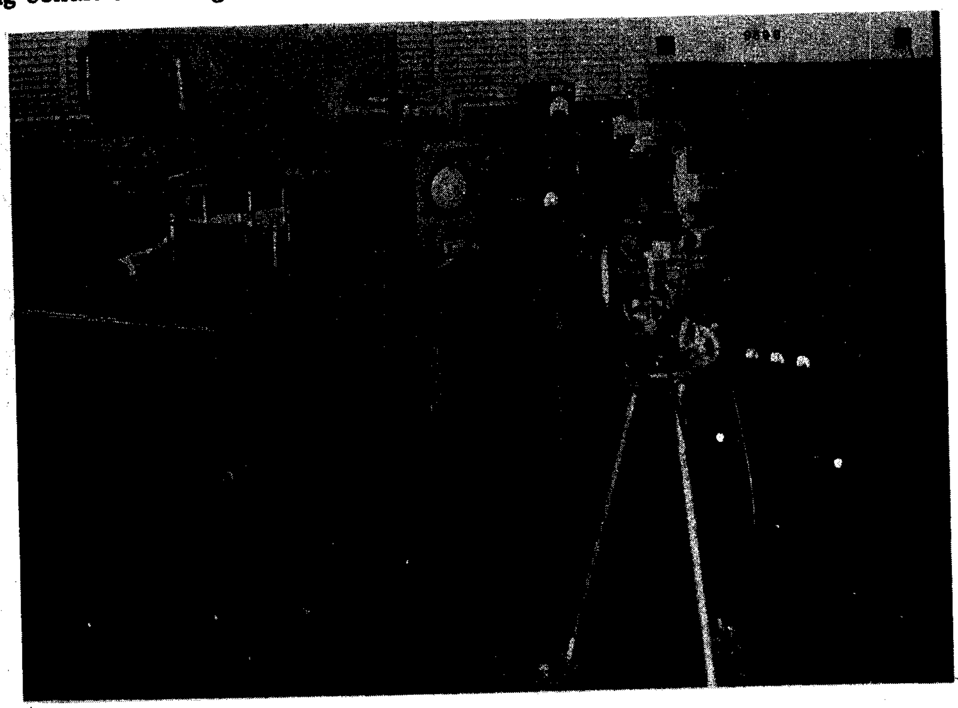

Figure 3 - Equipment for the measurement of polarization characteristics

values of θ and ϕ are controlled there. The receiving antenna, usually a small horn, is fastened on the special mount so that the axis of its main lobe, the axis of rotation of the horn, and the line of sight between the two antennas are coincident. The received signal is displayed on a PPI scope or on a meter. For qualitative measurements the horn is rotated rapidly enough to allow the display of the signal on the scope. The face of the scope may be fitted with a specially calibrated transparent chart for the measurement of the ellipticity e and the orientation angle ψ of the major axis of the polarization ellipse.

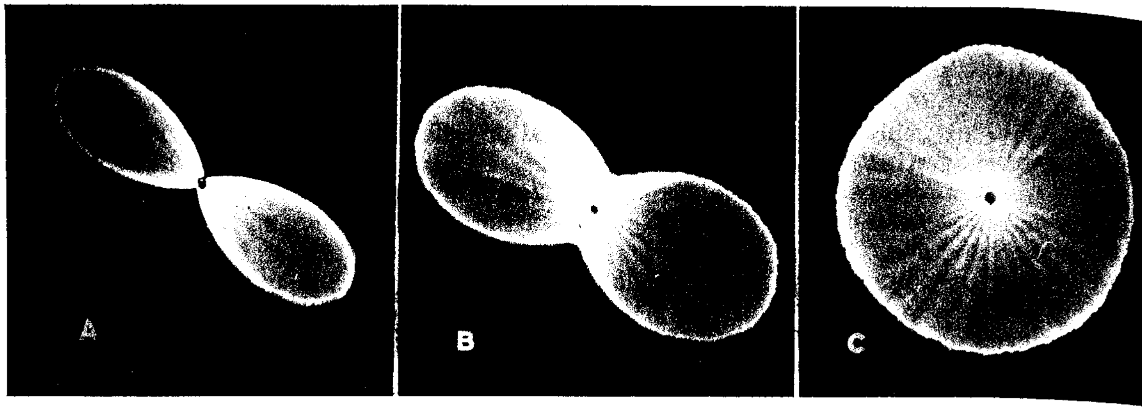

Figure 4 - Polarization patterns on the scope for:
(A) Linear polarization
(B) Elliptical polarization
(C) Circular polarization

Photographs of polarization patterns on the scope for linear, elliptical, and circular polarizations are shown in Figures 4A, B, and C. For more accurate work, the rotation is made slow enough to permit a meter to follow the signal as detected. It is then necessary to record only the ratio of the maximum signal to the minimum signal, and the value of ψ.

The equipment shown in Figure 3 is intended for use indoors for making measurements on low gain microwave radiators. For measurements outdoors on high gain antennas, the Antenna Research Branch finds it convenient to have the rotating linearly polarized antenna transmitting. Due to the large separation between transmitting and receiving sites, the orientation of polarization is controlled remotely at the receiving site.

Two Circularly Polarized Antennas - The second method utilizes the fact that an elliptically polarized field can be represented by two circularly polarized components of opposite sense of polarization. The antenna under test is transmitting, as before. Two circularly polarized antennas are used for receiving, and are assumed identical in their impedance and pattern characteristics except for sense of polarization. If the same receiver with a square law detector is used with both antennas, the received signals are proportional to the square of the quantities $|\mathcal{E}_R|$ and $|\mathcal{E}_L|$ which appear in Equation (10). The ellipticity may then be calculated from this equation. The sense of polarization will be right- or left-handed according as $|\mathcal{E}_R|$ is greater or less than $|\mathcal{E}_L|$.

For many applications, the two circularly polarized antennas could be two helices[2,3]

[2] Marston, A. E., and Adcock, M. D., "Radiation from Helices," NRL Report 3634 (Unclassified), March 8, 1950

[3] Kraus, J. D., and Williamson, J. C., "Characteristics of Helical Antennas Radiating in the Axial Mode," J. Applied Physics, 19:87-96, January 1948

in opposite senses. If desired, a single antenna[4,5] with two pairs of terminals for ...ring the right- and left-handed circular components of the incident field may be used.

Comparison of the Two Methods - The two methods do not yield directly the same ...mation, since the first does not give the sense of polarization, while the second does ...ive the orientation angle, ψ. With suitable modifications either method may be made ...eld all the required information, but the techniques will not be discussed here.

From patterns of the test antenna obtained by receiving separately on each of the two ...ularly polarized antennas, one may easily calculate not only the ellipticity but also ...radiation intensity as a function of direction. In general, however, it is much simpler ...btain the single linearly polarized antenna than to obtain the pair of right- and left-...ded circularly polarized antennas. Moreover, as will be shown in the following paragraphs, measurements of ellipticity with a linearly polarized antenna will, in general, be more accurate than measurements with a pair of circularly polarized antennas. Thus, for accuracy and for simplicity of instrumentation, the first method employing a linearly polarized antenna is preferable; for ease in obtaining ellipticity and radiation intensity as a function of direction, the second method is preferable provided the accuracy obtainable is sufficient.

In order to compare the accuracy of the two methods for measuring ellipticity, the following assumptions are made: (1) The existence of a cross-polarized component[6] in the receiving antennas, taken alternately to be a maximum of either 25 db, 30 db, or 40 db below the desired component; (2) a difference in antenna gain between the two "circularly" polarized antennas of at most 0.3 db; and (3) errors in meter reading of at most 0.1 db.

Details of the derivation of the formulas for errors in the measured value of ellipticity are given in Appendix II. The results are shown in Figure 5. The ellipticity in db as calculated from measurements is referenced on the horizontal scales. The maximum deviations of the true value of the ellipticity from a measured value, resulting from the three types of errors assumed above, are referenced on the vertical scales. The parameter for the two sets of curves is the db difference between the desired component and the cross-polarized component, referenced in Figure 5 as ρ. These values of the parameter are realistic, since the small horn and the two helices used as the receiving antennas in the two methods possessed cross-polarized components about 30 db down from the desired components.

To illustrate the use of the graph, assume that $\rho = 30$ db and that a value of e = 3 db is calculated from measurements on a test antenna. Then it is seen from Figure 5 that the true value of e lies between 2.6 db and 3.4 db if linear components are used and between 2.3 db and 3.8 db if circular components are used. It is obvious from Figure 5 that linear components give more accurate results than circular components for all values of e plotted, admitting the errors assumed above.

---

[4] Chait, H. N., "An Arbitrarily Polarized Antenna for use on X-Band," NRL Report R-3416, March 15, 1949

[5] Chait, H. N., "Microwave Antenna for Arbitrary Polarization," Electronics (process of publication)

[6] In the case of circular polarization, this is the component of the opposite sense.

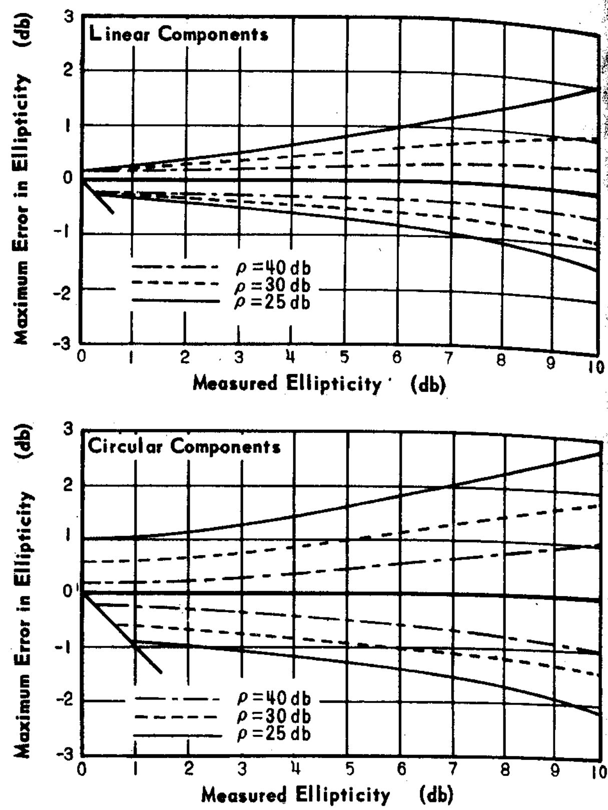

Figure 5 - Errors in measured ellipticity

Measurement of Gain

The distant field $\vec{\mathcal{E}}(P, t)$ of any antenna is given by Equation (14). Denoting by $\mathcal{E}_u$ and g_u the components of $\vec{\mathcal{E}}$ and $\vec{g}$ in the direction of an arbitrary complex unit vector $\vec{u}$, it follows from Equation (17) that

$$|\mathcal{E}_u|^2 = \sqrt{\mu/\epsilon}\; P_t |g_u|^2/2\pi R^2 \; .$$

Let $P_u(\theta, \phi) = \frac{1}{2}\sqrt{\epsilon/\mu}\; R^2 |\mathcal{E}_u|^2$ and $G_u(\theta, \phi) = 4\pi P_u(\theta, \phi)/P_t$ denote respectively the power per unit solid angle and the gain associated with the component $\mathcal{E}_u$. Hence

$$|g_u|^2 = 4\pi \; P_u(\theta, \phi)/P_t = G_u(\theta, \phi) \; .$$

If a complex unit vector $\vec{v}$ is chosen so that $\vec{u}$ and $\vec{v}$ are orthogonal, then

$$|\vec{g}|^2 = |g_u|^2 + |g_v|^2 \; .$$

By Equation (18), $|\vec{g}|^2 = G(\theta, \phi)$, where $G(\theta, \phi)$ is the power gain relative to an isotropic radiator in the direction (θ, ϕ). Hence

$$G(\theta, \phi) = G_u(\theta, \phi) + G_v(\theta, \phi) \quad , \tag{25}$$

and it is seen that the gain $G(\theta, \phi)$ is the sum of the gains of any two orthogonal components. Equation (25) may also be expressed in the form

$$G(\theta, \phi) = 4\pi \left\{ P_u(\theta, \phi) + P_v(\theta, \phi) \right\} /P_t = 4\pi P(\theta, \phi)/P_t \; .$$

The recommended procedure in the experimental determination of the power gain of an elliptically polarized antenna is summarized below. The simplest pair of orthogonal components is chosen, namely, linear. The test antenna whose gain function is to be determined is used as a receiving antenna.

1. Orient the linearly polarized transmitting antenna so that maximum signal is received for an arbitrary orientation of the test antenna.

2. Orient the test antenna so that the direction (θ, ϕ) in which the gain determination is desired coincides with the line of sight between the two antennas. This direction is usually chosen as the direction of maximum gain. The received signal is proportional to the gain of the linear component of the test antenna which is parallel to the E-vector of the transmitting antenna, assuming, of course, a square law detector.

3. Rotate either the transmitting or the receiving antenna exactly 90° about their line of sight. The received signal is proportional to the gain of the orthogonal component.

4. The power gain in the chosen direction is proportional to the sum of these two components. The gain may be determined absolutely by direct comparison with a calibrated linearly polarized gain standard in precisely the same manner as in the case of linearly polarized test antennas.

Definition of Beamwidth - When considering elliptically polarized antennas, one may be interested not only in the radiation pattern showing the "total" power per unit solid angle but also in patterns showing power per unit solid angle associated with a particular component. To any one of these patterns, the term beamwidth may be applied in the usual way. In general, of course, both the patterns and beamwidths will differ from component to component. Therefore, in speaking of "the" beamwidth of an elliptically polarized antenna, one should indicate clearly which pattern is referred to.

Measurement of Power Transfer

The equation for the power transfer between two elliptically polarized antennas can be obtained from Equation (21). This relation can be expressed in terms of measurable quantities by use of Equations (17), (18), (20), and (23) and the definition for S, so that

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2 \frac{(1+e_i^2)(1+e_r^2) \pm 4\, e_i e_r + (1-e_i^2)(1-e_r^2)\cos 2\psi}{2(1+e_i^2)\,(1+e_r^2)}, \tag{26}$$

where G_t and G_r are the power gain of the transmitting and receiving antennas respectively. The angle ψ as previously defined is the angle between the major axes of the polarization ellipses of the transmitting and receiving antennas. The sign of the term $4\, e_i e_r$ is chosen + or - according as the two antennas are polarized in the same or opposite sense.

In a previous section of this report, the problems of determining conditions for complete rejection of an incident field by a receiving antenna and for maximum acceptance of an incident field by a receiving antenna of constant receiving area were considered. It was shown that for these two cases the polarization efficiency f is respectively zero and unity. Hence for maximum power transfer between two antennas, Equation (26) reduces to

$$P_r = P_t G_t G_r (\lambda / 4\pi \ R)^2 \quad . \tag{27}$$

This occurs when the two antennas have the same ellipticity, the same sense of polarization, and the major axes of their polarization ellipses are parallel. Let us now consider the application of Equation (26) to other particular cases.

Case 1. Let both antennas be linearly polarized. Then $e_i = e_r = \infty$, and Equation (26) becomes

$$P_r = P_t G_t G_r \ (\lambda \cos \psi / 4 \pi \ R)^2 \ .$$

If the linear polarizations are parallel, then $\psi = 0$ and the expression for P_r simplifies to the familiar form of the maximum power transfer equation for linearly polarized antennas. If the linear polarizations are mutually perpendicular, then $\psi = \pi/2$ and $P_r = 0$ as expected. In any event, if all the variables are fixed except the relative orientation of the two linear polarizations, then P_r varies as $\cos^2 \psi$, a well known fact.

Case 2. Let both antennas be circularly polarized. Then $e_i = e_r = 1$. For the same sense of polarization the plus sign is taken in Equation (26) so that

$$P_r = P_t G_t G_r \ (\lambda / 4\pi \ R)^2 \quad .$$

Note that this expression is Equation (27). The result is a specific example of the general statement made previously concerning the expression of P_r for maximum power transfer.

If the antennas are circularly polarized in opposite senses, then the minus sign is chosen and $P_r = 0$, as expected.

Note that if one of the antennas is circularly polarized, the term in Equation (26) containing ψ becomes zero. This is obvious from the fact that at least one of the polarization ellipses has become a circle, so that relative orientation of polarization becomes meaningless.

Case 3. Let one of the antennas be circularly polarized and the other linearly polarized. Then either $e_i = 1$ and $e_r = \infty$, or $e_i = \infty$ and $e_r = 1$. In either case, Equation (26) reduces to

$$P_r = \frac{1}{2} P_t G_t G_r \ (\lambda / 4\pi \ R)^2 \quad .$$

Thus only one-half of the maximum possible power is received, assuming of course prescribed values of P_t, G_t, G_r, λ, and R.

* * *

APPENDIX I

Proof That: $\mathcal{V}_r = \mathcal{K}\, \vec{\mathcal{E}} \cdot \vec{h}^*$

It is assumed that the antenna in question is connected to the generator by means of a transmission line or waveguide which supports only a single propagating mode at the given frequency. At an arbitrary reference cross-section of the transmission line, voltage and current parameters may be introduced in the usual way. For the purpose of this discussion all circuit elements which are on the opposite side of the reference cross-section from the generator or receiver will be considered part of the antenna. It will further be assumed that the generator or receiver as the case may be is matched to the line, but that the line is not necessarily matched to the antenna in free space.

Introduce a rectangular coordinate system with origin 0 at an arbitrary point in front of the antenna with the z-axis in the direction of interest for transmission and reception. Let s_x denote the input voltage to the receiver when a uniform plane wave with its $\vec{E}$-vector parallel to the x-axis, and with unit amplitude at 0 is incident on the antenna in the direction of the z-axis. Let s_y be similarly defined. If an elliptically polarized plane wave represented by $\vec{\mathcal{E}} = \mathcal{E}_x \vec{i}_x + \mathcal{E}_y \vec{i}_y$ at 0 is incident on the antenna from the same direction, then by the superposition principle, the input voltage, $\mathcal{V}$, to the receiver is given by

$$\mathcal{V} = \mathcal{E}_x s_x + \mathcal{E}_y s_y \quad .$$

This may be written as $\mathcal{V} = \vec{\mathcal{E}} \cdot \vec{s}$ where $\vec{s} = s_x \vec{i}_x + s_y \vec{i}_y$.

Now let a second antenna be located at a distant point $0'$ on the z-axis. It will be assumed that the field transmitted by the second antenna in the direction from $0'$ to 0 is linearly polarized. It will further be assumed for this second antenna that both the antenna and generator or receiver are matched to a line which supports only a single propagating mode. In the same way as for the first antenna, a vector $\vec{s}$ may be associated with the second antenna to represent the voltage received by the second when an elliptically polarized plane wave is incident upon it in the direction from 0 to $0'$. Subscripts 1 and 2 will be used to distinguish between these two vectors. Two cases will now be considered. It will be assumed that antenna (2) is oriented so that the transmitted $\vec{E}$-vector is parallel to the x-axis in the first case, and parallel to the y-axis in the second case. The corresponding $\vec{s}$ vectors for antenna (2) in these two cases will be denoted by $\vec{s}_2 = s_2 \vec{i}_x$ and $\vec{s}_2' = s_2 \vec{i}_y$. Let $\vec{g}_2 = g_2 \vec{i}_x$ and $\vec{g}_2' = g_2 \vec{i}_y$ be the corresponding transmitting vectors for antenna (2). Since the orientation of antenna (1) is not changed, antenna (1) will be represented by $\vec{s}_1 = s_{1x} \vec{i}_x + s_{1y} \vec{i}_y$ and $\vec{g}_1 = g_{1x} \vec{i}_x + g_{1y} \vec{i}_y$.

Now let a unit voltage be impressed at the terminals of antenna (1), and let P_1 be the transmitted power. Then the input voltage to the receiver of antenna (2) is given by

$$\mathcal{V}_2 = \left(\sqrt{P_1} / (2\pi\sqrt{\epsilon/\mu}\,)^{\frac{1}{2}}\right)\left(e^{-jkR}/R\right) \vec{g}_1 \cdot \vec{s}_2$$

for the first orientation of antenna (2), and by

$$\mathcal{V}_2' = \left(\sqrt{P_1} / (2\pi\sqrt{\epsilon/\mu}\,)^{\frac{1}{2}}\right)\left(e^{-jkR}/R\right) \vec{g}_1 \cdot \vec{s}_2'$$

for the second. Let Γ be the reflection coefficient at the terminals of antenna (1) when transmitting, and let Z_1 be the impedance of the line. If $\rho = (1+\Gamma)/(1-\Gamma)$, and $r = \mathrm{Re}\,\rho$, then for unit input voltage at the terminals, $P_1 = r/2\,|\rho|^2 Z_1$. If we let $\mathcal{M} = (e^{-jkR}/R)/(4\pi\sqrt{\epsilon/\mu})^{\frac{1}{2}}$, then $\mathcal{V}_2$ and $\mathcal{V}_2'$ can be represented as follows:

$$\mathcal{V}_2 = \frac{\mathcal{M}}{|\rho|}\sqrt{\frac{r}{Z_1}}\; g_{1x}\; s_2\,,$$

$$\mathcal{V}_2' = \frac{\mathcal{M}}{|\rho|}\sqrt{\frac{r}{Z_1}}\; g_{1y}\; s_2\,.$$

Similarly if a unit voltage is applied to the terminals of antenna (2) for the two orientations of antenna (2), then the input voltages to the receiver of antenna (1) are given respectively by

$$\mathcal{V}_1 = \mathcal{M}\overleftarrow{g_2}\cdot\overleftarrow{s_1}/\sqrt{Z_2} = \mathcal{M}g_2\, s_{1x}/\sqrt{Z_2}\,,$$

$$\mathcal{V}_1' = \mathcal{M}\overleftarrow{g_2'}\cdot\overleftarrow{s_1}/\sqrt{Z_2} = \mathcal{M}g_2\, s_{1y}/\sqrt{Z_2}\,,$$

where Z_2 is the impedance of the transmission line for antenna (2). It should be remembered that these input voltages are the voltages that appear when the receivers have impedances which are equal to the respective line impedances.

The voltages $\mathcal{V}_2$, $\mathcal{V}_2'$, $\mathcal{V}_1$, and $\mathcal{V}_1'$ will now be calculated by another method. The terminal relations between antennas (1) and (2) can be represented for each of the two orientations of antenna (2) by means of a four-terminal network. Thus

$$\mathcal{V}_1 = Z_{11} I_1 - Z_{12} I_2\,,$$

$$\mathcal{V}_2 = Z_{12} I_1 - Z_{22} I_2\,;$$

and

$$\mathcal{V}_1' = Z_{11}' I_1' - Z_{12}' I_2'\,,$$

$$\mathcal{V}_2' = Z_{12}' I_1' - Z_{22}' I_2'\,.$$

It is obvious that as the separation R between the antenna increases, then

$$\lim_{R\to\infty} Z_{12} = \lim_{R\to\infty} Z_{12}' = 0$$

$$\lim_{R\to\infty} Z_{11} = \lim_{R\to\infty} Z_{11}' = \rho Z_1$$

$$\lim_{R\to\infty} Z_{22} = \lim_{R\to\infty} Z_{22}' = Z_2\,.$$

Let antenna (1) transmit and antenna (2) receive. We find by solving for $\mathcal{V}_2$ and $\mathcal{V}_2'$ when $\mathcal{V}_1 = \mathcal{V}_1' = 1$, and antenna (2) is terminated by an impedance, Z_2, that

$$\mathcal{V}_2 \doteq Z_{12}/2\rho Z_1$$

$$R \gg 1$$

$$\mathcal{V}_2' \doteq Z_{12}'/2\rho Z_1\,.$$

Similarly if antenna (2) transmits and antenna (1) receives, and antenna (1) is terminated by an impedance Z_1, then for unit input voltage to antenna (2),

$$\mathcal{V}_1 \doteq Z_{12}/(1+\rho)Z_2 \qquad R >> 1$$
$$\mathcal{V}_1' \doteq Z_{12}/(1+\rho)Z_2 \quad .$$

Combining results, we find that for large R,

$$\mathcal{M}(1+\rho)\sqrt{Z_2}\;\mathcal{g}_2 \mathcal{A}_{1x} = Z_{12} = 2\mathcal{M}\frac{\rho}{|\rho|}\sqrt{rZ_1}\;\mathcal{g}_{1x}\mathcal{A}_2 ,$$
$$\mathcal{M}(1+\rho)\sqrt{Z_2}\;\mathcal{g}_2 \mathcal{A}_{1y} = Z_{12}' = 2\mathcal{M}\frac{\rho}{|\rho|}\sqrt{rZ_1}\;\mathcal{g}_{1y}\mathcal{A}_2 . \tag{28}$$

Thus $\mathcal{A}_{1x}/\mathcal{A}_{1y} = \mathcal{g}_{1x}/\mathcal{g}_{1y}$, from which it follows that $\vec{\mathcal{A}}_1 = \mathcal{T}\,\vec{\mathcal{g}}_1$, where $\mathcal{T}$ is a constant. From Equations (28) we conclude that

$$(1+\rho)\sqrt{Z_2}\;\mathcal{g}_2\vec{\mathcal{A}}_1 = 2\frac{\rho}{|\rho|}\sqrt{rZ_1}\,\mathcal{A}_2\,\vec{\mathcal{g}}_1 \quad .$$

Hence $(1+\rho)\sqrt{Z_2}\;\mathcal{T}\mathcal{g}_2 = 2\frac{\rho}{|\rho|}\sqrt{rZ_1}\;\mathcal{A}_2$,

and
$$|\mathcal{T}| = 2\sqrt{\frac{rZ_1}{Z_2}}\;\frac{|\mathcal{A}_2|}{|1+\rho||\mathcal{g}_2|} \quad .$$

Since the field transmitted by antenna (2) is linearly polarized, it follows by a well known theorem that if a linearly polarized plane wave represented by $\vec{\mathcal{E}}$ is incident upon it, and if $\vec{\mathcal{E}}$ is parallel to $\vec{\mathcal{g}}_2$, then the received power is given by

$$P_r = \frac{1}{2}\sqrt{\epsilon/\mu}\,|\vec{\mathcal{E}}|^2|\mathcal{g}_2|^2\;\lambda^2/4\pi \quad .$$

On the other hand from the definition of $\mathcal{A}_2$, $|\mathcal{V}_r| = |\vec{\mathcal{E}}\cdot\vec{\mathcal{A}}_2| = |\vec{\mathcal{E}}||\mathcal{A}_2|$.

Hence $P_r = \frac{1}{2}|\mathcal{E}|^2|\mathcal{A}_2|^2/Z_2$, and $|\mathcal{A}_2|/|\mathcal{g}_2| = \lambda\sqrt{Z_2}/(4\pi\sqrt{\mu/\epsilon})^{\frac{1}{2}}$.

Substituting in the expression for $|\mathcal{T}|$ we find

$$|\mathcal{T}| = \frac{2\lambda}{|1+\rho|}\sqrt{\frac{rZ_1}{4\pi\sqrt{\mu/\epsilon}}} \quad .$$

Now $1+\rho = 2/(1-\Gamma)$, and $r = \mathrm{Re}(1+\Gamma)/(1-\Gamma) = (1-|\Gamma|^2)/|1-\Gamma|^2$.

Hence
$$|\mathcal{T}| = \lambda\sqrt{\frac{Z_1(1-|\Gamma|^2)}{4\pi\sqrt{\mu/\epsilon}}} \quad .$$

Now let $\vec{\mathcal{L}} = \lambda\vec{\mathcal{g}}_1^{\,*}/\sqrt{4\pi}$. Then if an elliptically polarized field $\vec{\mathcal{E}}$ is incident on antenna (1), the received voltage $\mathcal{V}_r$ is given by

$$\mathcal{V}_r = \vec{\mathcal{E}} \cdot \vec{s}_1 = \mathcal{T} \vec{\mathcal{E}} \cdot \vec{g}_1 = \mathcal{T} \frac{\sqrt{4\pi}}{\lambda} \quad \vec{\mathcal{E}} \cdot \vec{\mathcal{L}}^* \quad .$$

But

$$\left| \frac{\mathcal{T}\sqrt{4\pi}}{\lambda} \right| = \left\{ \sqrt{\epsilon/\mu} \; Z_1(1 - |\Gamma|^2) \right\}^{\frac{1}{2}} \quad .$$

This concludes the proof of the theorem.

* * *

APPENDIX II

Derivation of Errors in Measured Ellipticity

Linearly Polarized Receiving Antenna

Consider first the possible error in the measured ellipticity if the receiving antenna, which is presumed to be linearly polarized, is not truly linear. Let,

$$e_r = 20 \log |\tan \alpha_r|$$

and

$$e_i = 20 \log \tan \alpha_i$$

denote respectively the ellipticities in db of the receiving antenna and of the test antenna whose ellipticity is being measured. Let $\alpha_r = \pi/2 + \delta_r$, where $-\delta \leq \delta_r \leq \delta$, and $\pi/4 \leq \alpha_i < \pi/2$. The angle δ will determine the maximum deviation of the receiving antenna from linearity, and the values of δ considered are small.

Since the receiving antenna is presumed to be linearly polarized, the ellipticity is calculated from the ratio of the maximum to the minimum values of the power received. From Equation (24), this ratio is

$$\frac{f_{max}}{f_{min}} = \frac{\sin^2(\alpha_i - \delta_r)}{\cos^2(\alpha_i + \delta_r)}$$

$$= \tan^2 \alpha_i \left(\frac{1 - \cot \alpha_i \tan \delta_r}{1 - \tan \alpha_i \tan \delta_r} \right)^2 .$$

The function on the right-hand side has a maximum value when $\delta_r = \delta$ and a minimum when $\delta_r = -\delta$.

If we now assume in addition a maximum error of ϵ db in any two readings due to instrumentation errors, and if we let e represent the measured ellipticity in db, then

$$e_{max} = 20 \log \tan \alpha_i + 20 \log \left(\frac{1 - \cot \alpha_i \tan \delta}{1 - \tan \alpha_i \tan \delta} \right) + \epsilon$$

$$\cong e_i + 8.69 \tan \delta (\tan \alpha_i - \cot \alpha_i) + \epsilon ,$$

provided $\tan \alpha_i \tan \delta << 1$. Similarly

$$e_{min} = e_i - 8.69 \tan \delta (\tan \alpha_i - \cot \alpha_i) - \epsilon .$$

For given values of δ and ϵ, e_{max} and e_{min} can be plotted as functions of the true ellipticity e_i. Then by graphical inversion, the range of possible values of e_i for a given measured value of e can be determined.

Circularly Polarized Receiving Antennas

Consider now the possible error in the measured ellipticity if a pair of receiving antennas are used which are presumed to be right- and left-handed circularly polarized but which are not truly circular. Let e_R, e_L and e_i denote respectively the ellipticities in db of the right- and left-handed circularly polarized receiving antennas and of the test antenna.

Let $e_i = 20 \log |\tan \alpha_i|$, $e_R = 20 \log |\tan \alpha_R|$, $e_L = 20 \log |\tan \alpha_L|$.

Finally, let $\alpha_i = \pi/4 + \delta_i$, $\alpha_R = \pi/4 + \delta_R$, $\alpha_L = 3\pi/4 - \delta_L$, where $0 \leq {\delta_R \atop \delta_L} < \delta$.

This time the angle δ is a measure of the greatest deviation from circularity of the receiving antennas.

If P_R and P_L denote the powers received by the right- and left-handed circularly polarized antennas respectively, then the measured value of e in db is found from Equation (10) to be

$$e = 20 \log \tan \left| \frac{1 + \sqrt{P_R/P_L}}{1 - \sqrt{P_R/P_L}} \right| .$$

Thus to find the deviation of the measured value of e from the true value e_i, it is necessary to investigate the deviation of the measured value of P_R/P_L from the true value that would be obtained if there were no sources of error.

Let f_R and f_L denote the polarization efficiencies corresponding to the right- and left-handed receiving antennas, and let h_R and h_L denote the magnitudes of their respective reception vectors. Then if there were no instrumentation errors, the ratio P_R/P_L would be given by

$$P_R/P_L = h_R^2 \ f_R/h_L^2 \ f_L \tag{29}$$

Let us now determine the maximum and minimum values of the quantity on the right of Equation (29). If only the orientation of the receiving antennas are considered, then by Equation (24)

$$f_{R\ max} = \cos^2(\alpha_i - \alpha_R) = \cos^2(\delta_i - \delta_R) ,$$

$$f_{R\ min} = \sin^2(\alpha_i + \alpha_R) = \cos^2(\delta_i + \delta_R) ,$$

$$f_{L\ max} = \cos^2(\alpha_i - \alpha_L) = \sin^2(\delta_i + \delta_L) ,$$

$$f_{L\ min} = \sin^2(\alpha_i + \alpha_L) = \sin^2(\delta_i - \delta_L) .$$

At this point it shall be assumed for simplicity that $\delta_i > \delta > {\delta_R \atop \delta_L}$.

This means that the antenna under test is not more nearly circular than either of the receiving antennas. (In preparing Figure 6 this restriction was dropped.) Under the

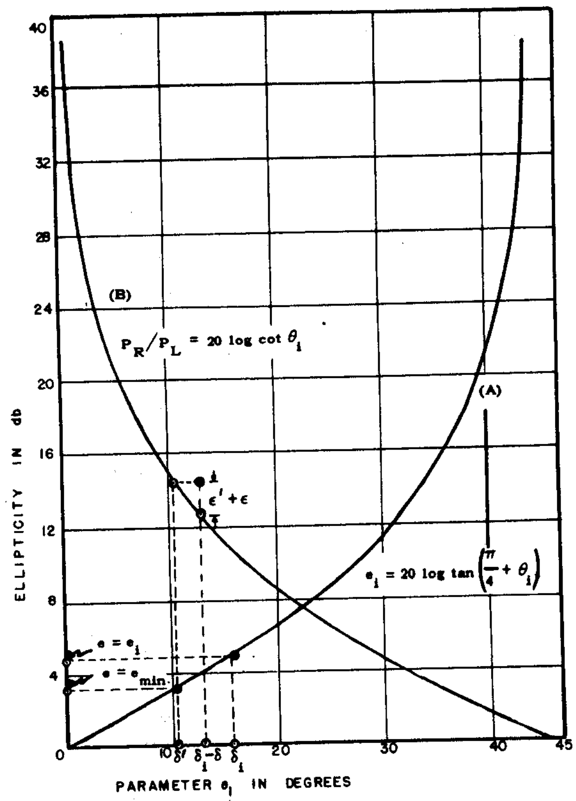

Figure 6 - Ellipticity as a function of errors using circular components

above assumption, if the variation of the ratio f_R/f_L with δ_R and δ_L is also included, it can be shown that

$$\left.\begin{aligned} \frac{f_R}{f_L} &\leqq \frac{\cos^2(\delta_i - \delta_R)}{\sin^2(\delta_i - \delta_L)} \leqq \frac{\cos^2(\delta_i - \delta)}{\sin^2(\delta_i - \delta)} = \cot^2(\delta_i - \delta) \\ \text{and} \qquad \frac{f_R}{f_L} &\geqq \frac{\cos^2(\delta_i + \delta_R)}{\sin^2(\delta_i + \delta_L)} \geqq \frac{\cos^2(\delta_i + \delta)}{\sin^2(\delta_i + \delta)} = \cot^2(\delta_i + \delta) \end{aligned}\right\} \qquad (30)$$

Now let ϵ' denote the db difference in gain between the two receiving antennas, and let ϵ denote, as before, the maximum difference in db of any two readings due to instrumentation errors. If $(P_R/P_L)_{max}$ and $(P_R/P_L)_{min}$ denote the maximum and minimum possible measured values of the ratio P_R/P_L, then it follows from Equations (29) and (30) that

$$10 \log (P_R/P_L)_{max} = 20 \log \cot (\delta_i - \delta) + \epsilon' + \epsilon$$

and

$$10 \log (P_R/P_L)_{min} = 20 \log \cot (\delta_i + \delta) - \epsilon' - \epsilon .$$

On the other hand, if no errors are present,

$$10 \log (P_R/P_L) = 20 \log \cot \delta_i .$$

Since

$$e_i = 20 \log \tan (\frac{\pi}{4} + \delta_i) ,$$

it is seen that e_i and P_R/P_L are simply related with the aid of the parameter δ_i. Hence if a number δ' is determined such that $20 \log \cot (\delta_i - \delta) + \epsilon' + \epsilon = 20 \log \cot \delta'$, then the minimum measured value of ellipticity is given by

$$e_{min} = 20 \log \tan (\frac{\pi}{4} + \delta') .$$

In a similar manner e_{max} may be determined.

With the aid of the above equations, e_{max} and e_{min} may be determined for any given value of e_i by a simple graphical procedure as illustrated in Figure 6. Find the point on curve (A) for which $e = e_i$. The abscissa of this point is δ_i. Now find the point on curve (B) whose abscissa is $\delta_i - \delta$. Increase the ordinate of this point by an amount $\epsilon' + \epsilon$ and find the point on curve (B) which has this new ordinate. The abscissa of this point is δ'. Finally e_{min} is obtained from curve (A) by finding the point on it with abscissa δ'. A similar procedure determines e_{max}.

Thus, as was the case in the first method using linear polarization, the maximum and minimum possible measured values of ellipticity are obtained in terms of the true ellipticity. Again in the same way one can obtain from these results the range within which the true ellipticity lies when the measured value of ellipticity is given.

* * *

NAVY-DPPO PRNC. WASH.